

An Interpolation Theorem for Sublinear Operators on Non-homogeneous Metric Measure Spaces

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Abstract. Let (\mathcal{X}, d, μ) be a metric measure space and satisfy the so-called upper doubling condition and the geometrically doubling condition. In this paper, the authors establish an interpolation result that a sublinear operator which is bounded from the Hardy space $H^1(\mu)$ to $L^{1,\infty}(\mu)$ and from $L^\infty(\mu)$ to the BMO-type space $\text{RBMO}(\mu)$ is also bounded on $L^p(\mu)$ for all $p \in (1, \infty)$. This extension is not completely straightforward and improves the existing result.

1 Introduction

Spaces of homogeneous type were introduced by Coifman and Weiss [3] as a general framework in which many results from real and harmonic analysis on Euclidean spaces have their natural extensions; see, for example, [4, 5, 6]. Recall that a metric space (\mathcal{X}, d) equipped with a nonnegative Borel measure μ is called a *space of homogeneous type* if (\mathcal{X}, d, μ) satisfies the following *measure doubling condition* that there exists a positive constant C_μ , depending on μ , such that for any ball $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$0 < \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)). \quad (1.1)$$

The measure doubling condition (1.1) plays a key role in the classical theory of Calderón-Zygmund operators. However, recently, many classical results concerning the theory of Calderón-Zygmund operators and function spaces have been proved still valid if the doubling condition is replaced by a less demanding condition such as the polynomial growth condition; see, for example [14, 16, 17, 15, 18] and the references therein. In particular, let μ be a non-negative Radon measure on \mathbb{R}^n which only satisfies the *polynomial growth condition* that there exist positive constants C and $\kappa \in (0, n]$ such that for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$\mu(\{y \in \mathbb{R}^n : |x - y| < r\}) \leq Cr^\kappa. \quad (1.2)$$

Such a measure does not need to satisfy the doubling condition (1.1). We mention that the analysis with non-doubling measures played a striking role in solving the long-standing open Painlevé's problem by Tolsa in [18].

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Because measures satisfying (1.2) are only different, not more general than measures satisfying (1.1), the Calderón-Zygmund theory with non-doubling measures is not in all respects a generalization of the corresponding theory of spaces of homogeneous type. To include the spaces of homogeneous type and Euclidean spaces with a non-negative Radon measure satisfying a polynomial growth condition, Hytönen [8] introduced a new class of metric measure spaces which satisfy the so-called upper doubling condition and the geometrically doubling condition (see, respectively, Definitions 2.1 and 2.2 below), and a notion of the regularized BMO space, namely, $\text{RBMO}(\mu)$ (see Definition 2.4 below). Since then, more and more papers focus on this new class of spaces; see, for example [11, 12, 10, 1, 9, 7, 13].

Let (\mathcal{X}, d, μ) be a metric measure space satisfying the upper doubling condition and the geometrically doubling condition. In [10], the atomic Hardy space $H^1(\mu)$ (see Definition 2.5 below) was studied and the duality between $H^1(\mu)$ and $\text{RBMO}(\mu)$ of Hytönen was established. Some of results in [10] were also independently obtained by Anh and Duong [1] via different approaches. Moreover, Anh and Duong [1, Theorem 6.4] established an interpolation result that a linear operator which is bounded from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $\text{RBMO}(\mu)$ is also bounded on $L^p(\mu)$ for all $p \in (1, \infty)$. The purpose of this paper is to generalize and improve the interpolation result for linear operators in [1] to sublinear operators in the current setting (\mathcal{X}, d, μ) , which is stated as follows.

Theorem 1.1. *Let T be a sublinear operator that is bounded from $L^\infty(\mu)$ to $\text{RBMO}(\mu)$ and from $H^1(\mu)$ to $L^{1,\infty}(\mu)$. Then T extends boundedly to $L^p(\mu)$ for every $p \in (1, \infty)$.*

In Section 2, we collect preliminaries we need. In Section 3, for $r \in (0, 1)$, we first show that the maximal function $M_r^\sharp(f)$, which is a variant of the sharp maximal function $M^\sharp(f)$ in [1], is bounded from $\text{RBMO}(\mu)$ to $L^\infty(\mu)$, then we establish a weak type estimate between the doubling maximal function $N(f)$ and $M^\sharp(f)$, and we also establish a weak type estimate for $N_r(f)$ with $r \in (0, 1)$, a variant of $N(f)$. Using these results we establish Theorem 1.1. We remark that the method for the proof of Theorem 1.1 is different from that of [1, Theorem 6.4]. Precisely, in the proof of [1, Theorem 6.4], the fact that the composite operator $M^\sharp \circ T$ of the sharp maximal function M^\sharp and a linear operator T is a sublinear operator was used. However, as far as we know, when T is sublinear, whether the composite operator $M^\sharp \circ T$ is a sublinear operator is unclear and so the proof of [1, Theorem 6.4] is not available.

Throughout this paper, we denote by C a positive constant which is independent of the main parameters involved but may vary from line to line. The subscripts of a constant indicate the parameters it depends on. The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq Cg$. Also, for a μ -measurable set E , χ_E denotes its characteristic function.

2 Preliminaries

In this section, we will recall some necessary notions and notation and the Calderón-Zygmund decomposition which was established in [1]. We begin with the definition of upper doubling space in [8].

Definition 2.1. A metric measure space (\mathcal{X}, d, μ) is called *upper doubling* if μ is a Borel measure on \mathcal{X} and there exists a dominating function $\lambda : \mathcal{X} \times (0, \infty) \rightarrow (0, \infty)$ and a positive constant C_λ such that for each $x \in \mathcal{X}$, $r \rightarrow \lambda(x, r)$ is non-decreasing, and for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2).$$

Remark 2.1. (i) Obviously, a space of homogeneous type is a special case of the upper doubling spaces, where one can take the dominating function $\lambda(x, r) \equiv \mu(B(x, r))$. Moreover, let μ be a non-negative Radon measure on \mathbb{R}^n which only satisfies the polynomial growth condition (1.2). By taking $\lambda(x, r) \equiv Cr^\kappa$, we see that $(\mathbb{R}^n, |\cdot|, \mu)$ is also an upper doubling measure space.

(ii) It was proved in [10] that there exists a dominating function $\tilde{\lambda}$ related to λ satisfying the property that there exists a positive constant $C_{\tilde{\lambda}}$ such that $\tilde{\lambda} \leq \lambda$, $C_{\tilde{\lambda}} \leq C_\lambda$, and for all $x, y \in \mathcal{X}$ with $d(x, y) \leq r$,

$$\tilde{\lambda}(x, r) \leq C_{\tilde{\lambda}} \tilde{\lambda}(y, r). \quad (2.1)$$

Based on this, in this paper, we *always assume* that the dominating function λ also satisfies (2.1).

Throughout the whole paper, we also *always assume* that the underlying metric space (\mathcal{X}, d) satisfies the following geometrically doubling condition introduced in [8].

Definition 2.2. A metric space (\mathcal{X}, d) is called *geometrically doubling* if there exists some $N_0 \in \mathbb{N}^+ \equiv \{1, 2, \dots\}$ such that for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0 .

The following coefficients $\delta(B, S)$ for all balls B and S were introduced in [8] as analogues of Tolsa's numbers $K_{Q,R}$ in [16]; see also [10].

Definition 2.3. For all balls $B \subset S$, let

$$\delta(B, S) \equiv \int_{(2S) \setminus B} \frac{d\mu(x)}{\lambda(c_B, d(x, c_B))}.$$

where and in that follows, for a ball $B \equiv B(c_B, r_B)$ and $\rho \in (0, \infty)$, $\rho B \equiv B(c_B, \rho r_B)$.

In what follows, for each $p \in (0, \infty)$, $L_{\text{loc}}^p(\mu)$ denotes the *set of all functions f such that $|f|^p$ is μ -locally integrable*.

Definition 2.4. Let $\eta \in (1, \infty)$ and $p \in (0, \infty)$. A function $f \in L_{\text{loc}}^p(\mu)$ is said to be in the *space* $\text{RBMO}_\eta^p(\mu)$ if there exists a non-negative constant C and a complex number f_B for any ball B such that for all balls B ,

$$\frac{1}{\mu(\eta B)} \int_B |f(y) - f_B|^p d\mu(y) \leq C^p$$

and that for all balls $B \subset S$,

$$|f_B - f_S| \leq C[1 + \delta(B, S)].$$

Moreover, the $\text{RBMO}_\eta^p(\mu)$ *norm* of f is defined to be the minimal constant C as above and denoted by $\|f\|_{\text{RBMO}_\eta^p(\mu)}$.

When $p = 1$, we write $\text{RBMO}_\eta^1(\mu)$ simply by $\text{RBMO}(\mu)$, which was introduced by Hytönen in [8]. Moreover, the spaces $\text{RBMO}_\eta^p(\mu)$ and $\text{RBMO}(\mu)$ coincide with equivalent norms, which is the special case of [7, Corollary 2.1].

Proposition 2.1. *Let $\eta \in (1, \infty)$ and $p \in (0, \infty)$. The spaces $\text{RBMO}_\eta^p(\mu)$ and $\text{RBMO}(\mu)$ coincide with equivalent norms.*

Remark 2.2. It was proved in [8, Lemma 4.6] that the space $\text{RBMO}(\mu)$ is independent of the choice of η . By this and Proposition 2.1, it is obvious that the space $\text{RBMO}_\eta^p(\mu)$ is independent of the choice of η .

We now recall the definition of the atomic Hardy space introduced in [10]; see also [1].

Definition 2.5. Let $\rho \in (1, \infty)$ and $p \in (1, \infty]$. A function $b \in L^1(\mu)$ is called a $(p, 1)_\lambda$ -atomic block if

- (i) there exists some ball B such that $\text{supp}(b) \subset B$;
- (ii) $\int_{\mathcal{X}} b(x) d\mu(x) = 0$;
- (iii) for $j = 1, 2$, there exist functions a_j supported on balls $B_j \subset B$ and $\lambda_j \in \mathbb{C}$ such that

$$b = \lambda_1 a_1 + \lambda_2 a_2,$$

and

$$\|a_j\|_{L^p(\mu)} \leq [\mu(\rho B_j)]^{1/p-1} [1 + \delta(B_j, B)]^{-1}.$$

Moreover, let

$$|b|_{H_{atb}^{1,p}(\mu)} \equiv |\lambda_1| + |\lambda_2|.$$

A function $f \in L^1(\mu)$ is said to belong to the *atomic Hardy space* $H_{atb}^{1,p}(\mu)$ if there exist $(p, 1)_\lambda$ -atomic blocks $\{b_j\}_{j \in \mathbb{N}}$ such that $f = \sum_{j=1}^{\infty} b_j$ and $\sum_{j=1}^{\infty} |b_j|_{H_{atb}^{1,p}(\mu)} < \infty$. The *norm* of f in $H_{atb}^{1,p}(\mu)$ is defined by

$$\|f\|_{H_{atb}^{1,p}(\mu)} \equiv \inf \left\{ \sum_j |b_j|_{H_{atb}^{1,p}(\mu)} \right\},$$

where the infimum is taken over all the possible decompositions of f as above.

Remark 2.3. It was proved in [10] that for each $p \in (1, \infty]$, the atomic Hardy space $H_{atb}^{1,p}(\mu)$ is independent of the choice of ρ , and that for all $p \in (1, \infty)$, the spaces $H_{atb}^{1,p}(\mu)$ and $H_{atb}^{1,\infty}(\mu)$ coincide with equivalent norms. Thus, in the following, we denote $H_{atb}^{1,p}(\mu)$ simply by $H^1(\mu)$.

At the end of this section, we recall the (α, β) -doubling property of some balls and the Calderón-Zygmund decomposition established by Anh and Duong [1, Theorem 6.3].

Given $\alpha, \beta \in (1, \infty)$, a ball $B \subset \mathcal{X}$ is called (α, β) -doubling if $\mu(\alpha B) \leq \beta \mu(B)$. It was proved in [8] that if a metric measure space (\mathcal{X}, d, μ) is upper doubling and $\beta > C_\lambda^{\log_2 \alpha} \equiv$

α^ν , then for every ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$ such that $\alpha^j B$ is (α, β) -doubling. Moreover, let (\mathcal{X}, d) be geometrically doubling, $\beta > \alpha^n$ with $n \equiv \log_2 N_0$ and μ a Borel measure on \mathcal{X} which is finite on bounded sets. Hytönen [8] also showed that for μ -almost every $x \in \mathcal{X}$, there exist arbitrarily small (α, β) -doubling balls centered at x . Furthermore, the radius of these balls may be chosen to be of the form $\alpha^{-j}r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0, \infty)$. Throughout this paper, for any $\alpha \in (1, \infty)$ and ball B , \tilde{B}^α denotes the *smallest* (α, β_α) -doubling ball of the form $\alpha^j B$ with $j \in \mathbb{Z}_+$, where

$$\beta_\alpha \equiv \max\{\alpha^n, \alpha^\nu\} + 30^n + 30^\nu = \alpha^{\max\{n, \nu\}} + 30^n + 30^\nu. \quad (2.2)$$

Lemma 2.1. *Let $p \in [1, \infty)$, $f \in L^p(\mu)$ and $\ell \in (0, \infty)$ ($\ell > \ell_0 \equiv \gamma_0 \|f\|_{L^p(\mu)} / \mu(\mathcal{X})$) if $\mu(\mathcal{X}) < \infty$, where γ_0 is any fixed positive constant satisfying that $\gamma_0 > \max\{C_\lambda^{3 \log_2 6}, 6^{3n}\}$, C_λ is as in (2.2) and $n = \log_2 N_0$). Then*

(i) *there exists an almost disjoint family $\{6B_j\}_j$ of balls such that $\{B_j\}_j$ is pairwise disjoint,*

$$\begin{aligned} \frac{1}{\mu(6^2 B_j)} \int_{B_j} |f(x)|^p d\mu(x) &> \frac{\ell^p}{\gamma_0} \quad \text{for all } j, \\ \frac{1}{\mu(6^2 \eta B_j)} \int_{\eta B_j} |f(x)|^p d\mu(x) &\leq \frac{\ell^p}{\gamma_0} \quad \text{for all } j \text{ and all } \eta > 1, \end{aligned}$$

and

$$|f(x)| \leq \ell \quad \text{for } \mu - \text{almost every } x \in \mathcal{X} \setminus (\cup_j 6B_j);$$

(ii) *for each j , let S_j be a $(3 \times 6^2, C_\lambda^{\log_2(3 \times 6^2)+1})$ -doubling ball concentric with B_j satisfying that $r_{S_j} > 6^2 r_{B_j}$, and $\omega_j \equiv \chi_{6B_j} / (\sum_k \chi_{6B_k})$. Then there exists a family $\{\varphi_j\}_j$ of functions such that for each j , $\text{supp}(\varphi_j) \subset S_j$, φ_j has a constant sign on S_j and*

$$\begin{aligned} \int_{\mathcal{X}} \varphi_j(x) d\mu(x) &= \int_{6B_j} f(x) \omega_j(x) d\mu(x), \\ \sum_j |\varphi_j(x)| &\leq \gamma \ell \quad \text{for } \mu - \text{almost every } x \in \mathcal{X}, \end{aligned}$$

where γ is some positive constant depending only on (\mathcal{X}, μ) , and there exists a positive constant C , independent of f, ℓ and j , such that

$$\|\varphi_j\|_{L^\infty(\mu)} \mu(S_j) \leq C \int_{\mathcal{X}} |f(x) \omega_j(x)| d\mu(x),$$

and if $p \in (1, \infty)$,

$$\left\{ \int_{S_j} |\varphi_j(x)|^p d\mu(x) \right\}^{1/p} [\mu(S_j)]^{1/p'} \leq \frac{C}{\ell^{p-1}} \int_{\mathcal{X}} |f(x) \omega_j(x)|^p d\mu(x);$$

(iii) *if for any j , choosing S_j in (ii) to be the smallest $(3 \times 6^2, C_\lambda^{\log_2(3 \times 6^2)+1})$ -doubling ball of $(3 \times 6^2)B_j$, then $h \equiv \sum_j (f\omega_j - \varphi_j) \in H_{atb}^{1,p}(\mu)$ and there exists a positive constant C , independent of f and ℓ , such that*

$$\|h\|_{H_{atb}^{1,p}(\mu)} \leq \frac{C}{\ell^{p-1}} \|f\|_{L^p(\mu)}^p.$$

3 Proof Theorem 1.1

To prove Theorem 1.1, we also need some maximal functions in [8, 1] as follows. Let $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathcal{X}$. The *doubling Hardy-Littlewood maximal function* $N(f)(x)$ and the *sharp maximal function* $M^\sharp(f)(x)$ are respectively defined by setting,

$$N(f)(x) \equiv \sup_{\substack{B \ni x \\ B(6, \beta_6)\text{-doubling}}} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$

and

$$\begin{aligned} M^\sharp(f)(x) &\equiv \sup_{B \ni x} \frac{1}{\mu(5B)} \int_B |f(y) - m_{\tilde{B}^6}(f)| d\mu(y) \\ &\quad + \sup_{\substack{x \in B \subset S \\ B, S(6, \beta_6)\text{-doubling}}} \frac{|m_B(f) - m_S(f)|}{1 + \delta(B, S)}, \end{aligned}$$

where for any $f \in L^1_{\text{loc}}(\mu)$ and ball B , $m_B(f)$ means its average over B , namely, $m_B(f) \equiv \frac{1}{\mu(B)} \int_B f(x) d\mu(x)$. It was showed in [9, Lemma 2.3] that for any $p \in [1, \infty)$, Nf is bounded from $L^p(\mu)$ to $L^{p, \infty}(\mu)$.

Lemma 3.1. *Let $f \in \text{RBMO}(\mu)$, $r \in (0, 1)$ and $M_r^\sharp(f) \equiv [M^\sharp(|f|^r)]^{1/r}$. Then we have $M_r^\sharp f \in L^\infty(\mu)$, and moreover,*

$$\|M_r^\sharp f\|_{L^\infty(\mu)} \lesssim \|f\|_{\text{RBMO}(\mu)}.$$

Proof. From Remark 2.2, we deduce that for any ball B ,

$$|f_{\tilde{B}^6} - m_{\tilde{B}^6}(f)| \leq \frac{1}{\mu(\tilde{B}^6)} \int_{\tilde{B}^6} |f(x) - f_{\tilde{B}^6}| d\mu(x) \lesssim \|f\|_{\text{RBMO}(\mu)}.$$

On the other hand, by Proposition 2.1 and Remark 2.2, we see that

$$\|f\|_{\text{RBMO}(\mu)} \sim \|f\|_{\text{RBMO}_5^r(\mu)}.$$

From these facts, it follows that

$$\begin{aligned} &\frac{1}{\mu(5B)} \int_B ||f(x)|^r - m_{\tilde{B}^6}(|f|^r)| d\mu(x) \\ &\leq \frac{1}{\mu(5B)} \int_B [||f(x)|^r - |m_{\tilde{B}^6}(f)|^r| + ||m_{\tilde{B}^6}(f)|^r - m_{\tilde{B}^6}(|f|^r)|] d\mu(x) \\ &\lesssim \frac{1}{\mu(5B)} \int_B |f(x) - f_B|^r d\mu(x) + |f_B - f_{\tilde{B}^6}|^r + |f_{\tilde{B}^6} - m_{\tilde{B}^6}(f)|^r \\ &\quad + \frac{1}{\mu(\tilde{B}^6)} \int_{\tilde{B}^6} |f(x) - f_{\tilde{B}^6}|^r d\mu(x) \\ &\lesssim \left[1 + \delta(B, \tilde{B}^6)\right]^r \|f\|_{\text{RBMO}(\mu)}^r \lesssim \|f\|_{\text{RBMO}(\mu)}^r, \end{aligned}$$

where the last inequality follows from the fact that $\delta(B, \tilde{B}^6) \lesssim 1$, which holds by [10, Lemma 2.1].

On the other hand, for any $(6, \beta_6)$ -doubling balls $B \subset S$,

$$\begin{aligned} |m_B(|f|^r) - m_S(|f|^r)| &\leq |m_B(|f|^r) - |f_B|^r| + ||f_B|^r - |f_S|^r| + ||f_S|^r - m_S(|f|^r)| \\ &\leq m_B(|f - f_B|^r) + |f_B - f_S|^r + m_S(|f - f_S|^r) \\ &\lesssim [1 + \delta(B, S)]^r \|f\|_{\text{RBMO}(\mu)}^r. \end{aligned}$$

Combining these two inequalities finishes the proof of Lemma 3.1. \square

Lemma 3.2. *Let $p \in [1, \infty)$ and $f \in L_{\text{loc}}^1(\mu)$ such that $\int_{\mathcal{X}} f(x) d\mu(x) = 0$ if $\mu(\mathcal{X}) < \infty$. If for any $R > 0$,*

$$\sup_{0 < \ell < R} \ell^p \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}) < \infty,$$

we then have

$$\sup_{\ell > 0} \ell^p \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}) \lesssim \sup_{\ell > 0} \ell^p \mu\left(\left\{x \in \mathcal{X} : M^\sharp(f)(x) > \ell\right\}\right).$$

Proof. Recall the λ -good inequality in [1] that for some fixed constant $\nu \in (0, 1)$ and all $\epsilon \in (0, \infty)$, there exists some $\delta > 0$ such that for any $\ell > 0$,

$$\mu\left(\left\{x \in \mathcal{X} : N(f)(x) > (1 + \epsilon)\ell, M^\sharp(f)(x) \leq \delta\ell\right\}\right) \leq \nu \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}).$$

From this, it then follows that for R large enough and any $\epsilon > 0$,

$$\begin{aligned} &\sup_{0 < \ell < R} \ell^p \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}) \\ &\leq \sup_{0 < \ell < R} [(1 + \epsilon)\ell]^p \mu(\{x \in \mathcal{X} : N(f)(x) > (1 + \epsilon)\ell\}) \\ &\leq \nu(1 + \epsilon)^p \sup_{0 < \ell < R} \ell^p \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}) \\ &\quad + (1 + \epsilon)^p \sup_{\ell > 0} \ell^p \mu\left(\left\{x \in \mathcal{X} : M^\sharp(f)(x) > \delta\ell\right\}\right). \end{aligned}$$

Choosing ϵ small enough such that $\nu(1 + \epsilon)^p < 1$, our assumption then implies that

$$\sup_{0 < \ell < R} \ell^p \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}) \lesssim \sup_{\ell > 0} \ell^p \mu\left(\left\{x \in \mathcal{X} : M^\sharp(f)(x) > \ell\right\}\right).$$

Letting $R \rightarrow \infty$ then leads to the conclusion, which completes the proof of Lemma 3.2. \square

Lemma 3.3. *Let $r \in (0, 1)$ and $N_r(f) \equiv [N(|f|^r)]^{1/r}$. Then for any $p \in [1, \infty)$, there exists a positive constant C , depending on r , such that for suitable function f and any $\ell > 0$,*

$$\mu(\{x \in \mathcal{X} : N_r(f)(x) > \ell\}) \leq C \ell^{-p} \sup_{\tau \geq \ell} \tau^p \mu(\{x \in \mathcal{X} : |f(x)| > \tau\}).$$

Proof. For each fixed $\ell > 0$ and function f , decompose f as

$$f(x) = f(x)\chi_{\{x \in \mathcal{X}: |f(x)| \leq \ell\}}(x) + f(x)\chi_{\{x \in \mathcal{X}: |f(x)| > \ell\}} \equiv f_1(x) + f_2(x).$$

By the boundedness of N from $L^p(\mu)$ to $L^{p,\infty}(\mu)$, we obtain that

$$\begin{aligned} \mu(\{x \in \mathcal{X} : N_r(f)(x) > 2^{1/r}\ell\}) &\leq \mu(\{x \in \mathcal{X} : N(|f_2|^r)(x) > \ell^r\}) \\ &\lesssim \ell^{-rp} \int_{\mathcal{X}} |f_2(x)|^{rp} d\mu(x) \\ &\lesssim \mu(\{x \in \mathcal{X} : |f(x)| > \ell\}) \\ &\quad + \ell^{-rp} \int_{\ell}^{\infty} \tau^{rp-1} \mu(\{x \in \mathcal{X} : |f(x)| > \tau\}) d\tau \\ &\lesssim \mu(\{x \in \mathcal{X} : |f(x)| > \ell\}) \\ &\quad + \ell^{-p} \sup_{\tau > \ell} \tau^p \mu(\{x \in \mathcal{X} : |f(x)| > \tau\}), \end{aligned}$$

which implies our desired result. \square

Proof of Theorem 1.1. By the Marcinkiewicz interpolation theorem, we only need to prove that for all $f \in L^p(\mu)$ with $p \in (1, \infty)$ and $\ell > 0$,

$$\mu(\{x \in \mathcal{X} : |Tf(x)| > \ell\}) \lesssim \ell^{-p} \|f\|_{L^p(\mu)}^p. \quad (3.1)$$

We further consider the following two cases.

Case (i) $\mu(\mathcal{X}) = \infty$. Let $L_{b,0}^\infty(\mu)$ be the space of bounded functions with bounded supports and

$$L_{b,0}^\infty(\mu) \equiv \left\{ f \in L_b^\infty(\mu) : \int_{\mathcal{X}} f(x) d\mu(x) = 0 \right\}.$$

Then in this case, $L_{b,0}^\infty(\mu)$ is dense in $L^p(\mu)$ for all $p \in (1, \infty)$. Let $r \in (0, 1)$ and $N_r(g) \equiv [N(|g|^r)]^{1/r}$ for any $g \in L_{\text{loc}}^r(\mu)$. Notice that $|Tf| \leq N_r(Tf)$ μ -almost everywhere on \mathcal{X} . Then by a standard density argument, to prove (3.1), it suffices to prove that for all $f \in L_{b,0}^\infty(\mu)$ and $p \in (1, \infty)$,

$$\sup_{\ell > 0} \ell^p \mu(\{x \in \mathcal{X} : N_r(Tf)(x) > \ell\}) \lesssim \|f\|_{L^p(\mu)}^p. \quad (3.2)$$

For each fixed $\ell > 0$, applying Lemma 2.1, we obtain that $f = g + h$, where h is as Lemma 2.1 and $g \equiv f - h$, such that

$$\|g\|_{L^\infty(\mu)} \lesssim \ell, \quad h \in H^1(\mu) \quad (3.3)$$

and

$$\|h\|_{H^1(\mu)} \lesssim \ell^{1-p} \|f\|_{L^p(\mu)}^p. \quad (3.4)$$

For each $r \in (0, 1)$, define $M_r^\sharp(f) \equiv \{M^\sharp(|f|^r)\}^{1/r}$. Then, (3.3) together with the boundedness of T from $L^\infty(\mu)$ to $\text{RBMO}(\mu)$ and Lemma 3.1 shows that the function $M_r^\sharp(Tg)$ is bounded by a multiple of ℓ . Hence, if c_0 is a sufficiently large constant, we have

$$\left\{ x \in \mathcal{X} : M_r^\sharp(Tg)(x) > c_0 \ell \right\} = \emptyset. \quad (3.5)$$

On the other hand, since both f and h belong to $H^1(\mu)$, we see that $g \in H^1(\mu)$ and

$$\|g\|_{H^1(\mu)} \leq \|f\|_{H^1(\mu)} + \|h\|_{H^1(\mu)} \lesssim \|f\|_{H^1(\mu)} + \ell^{1-p} \|f\|_{L^p(\mu)}^p.$$

By this together with the boundedness of T from $H^1(\mu)$ to $L^{1,\infty}(\mu)$ and Lemma 3.3, we have that for any $p \in (1, \infty)$ and $R > 0$,

$$\sup_{0 < \ell < R} \ell^p \mu(\{x \in \mathcal{X} : N_r(Tg)(x) > \ell\}) \lesssim \sup_{0 < \ell < R} \ell^{p-1} \sup_{\tau \geq \ell} \tau \mu(\{x \in \mathcal{X} : |Tg(x)| > \tau\}) < \infty.$$

From this, (3.5), Lemma 3.2 and the fact that $N_r \circ T$ is quasi-linear, we deduce that there exists a positive constant \tilde{C} such that

$$\begin{aligned} & \sup_{\ell > 0} \ell^p \mu\left(\left\{x \in \mathcal{X} : N_r(Tf)(x) > \tilde{C}c_0\ell\right\}\right) \\ & \leq \sup_{\ell > 0} \ell^p \mu(\{x \in \mathcal{X} : N_r(Tg)(x) > c_0\ell\}) + \sup_{\ell > 0} \ell^p \mu(\{x \in \mathcal{X} : N_r(Th)(x) > c_0\ell\}) \\ & \lesssim \sup_{\ell > 0} \ell^p \mu\left(\left\{x \in \mathcal{X} : M_r^\sharp(Tg)(x) > c_0\ell\right\}\right) + \sup_{\ell > 0} \ell^p \mu(\{x \in \mathcal{X} : N_r(Th)(x) > c_0\ell\}) \\ & \lesssim \sup_{\ell > 0} \ell^p \mu(\{x \in \mathcal{X} : N_r(Th)(x) > \ell\}). \end{aligned} \tag{3.6}$$

From the boundedness of N from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ and the boundedness of T from $H^1(\mu)$ to $L^{1,\infty}(\mu)$, it follows that

$$\begin{aligned} & \mu(\{x \in \mathcal{X} : N_r(Th)(x) > \ell\}) \\ & \leq \mu\left(\left\{x \in \mathcal{X} : N\left(|Th|^r \chi_{\{x \in \mathcal{X} : |(Th)(x)| > \ell/2^{\frac{1}{r}}\}}\right) > \frac{\ell^r}{2}\right\}\right) \\ & \lesssim \ell^{-r} \int_{\mathcal{X}} \left|(Th)(x) \chi_{\{x \in \mathcal{X} : |(Th)(x)| > \ell/2^{\frac{1}{r}}\}}(x)\right|^r d\mu(x) \\ & \lesssim \ell^{-r} \mu\left(\left\{x \in \mathcal{X} : |(Th)(x)| > \ell/2^{\frac{1}{r}}\right\}\right) \int_0^{\ell/2^{\frac{1}{r}}} s^{r-1} ds \\ & \quad + \ell^{-r} \int_{\ell/2^{\frac{1}{r}}}^\infty s^{r-1} \mu(\{x \in \mathcal{X} : |(Th)(x)| > s\}) ds \\ & \lesssim \mu\left(\left\{x \in \mathcal{X} : |(Th)(x)| > \ell/2^{\frac{1}{r}}\right\}\right) + \frac{1}{\ell} \sup_{s \geq \ell/2^{\frac{1}{r}}} s \mu(\{x \in \mathcal{X} : |(Th)(x)| > s\}) \\ & \lesssim \frac{\|h\|_{H^1(\mu)}}{\ell} \lesssim \ell^{-p} \|f\|_{L^p(\mu)}^p, \end{aligned}$$

which together with (3.6) yields (3.2).

Case (ii) $\mu(\mathcal{X}) < \infty$. In this case, assume that $f \in L_b^\infty(\mu)$. Notice that if $\ell \in (0, \ell_0]$, where ℓ_0 is as in Lemma 2.1, then (3.1) holds trivially. Thus, we only have to consider the case when $\ell > \ell_0$. Let $r \in (0, 1)$, $N_r(f)$ be as in Lemma 3.3 and M_r^\sharp as in Case (i). For each fixed $\ell > \ell_0$, applying Lemma 2.1, we obtain that $f = g + h$ with g and h satisfying

(3.3) and (3.4), which together with the boundedness of T from $L^\infty(\mu)$ to $\text{RBMO}(\mu)$ and Lemma 3.1 yields (3.5) for $M_r^\sharp(Tg)$. We now claim that

$$F \equiv \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |Tg(x)|^r d\mu(x) \lesssim \ell^r, \quad (3.7)$$

where the constant depends on $\mu(\mathcal{X})$ and r . In fact, since $\mu(\mathcal{X}) < \infty$, we regard \mathcal{X} as a ball. Then $g_0 \equiv g - \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} g(x) d\mu(x) \in H^1(\mu)$. On the other hand, $|T1|^r \in \text{RBMO}(\mu)$ because of the fact that $T1 \in \text{RBMO}(\mu)$ and Lemma 3.1. This together with $\mu(\mathcal{X}) < \infty$ implies that

$$\int_{\mathcal{X}} |T1(x)|^r d\mu(x) < \infty.$$

Then by the boundedness of T from $H^1(\mu)$ to $L^{1,\infty}(\mu)$ and (3.3), we have

$$\begin{aligned} \int_{\mathcal{X}} |Tg(x)|^r d\mu(x) &\leq \int_{\mathcal{X}} \left\{ |Tg_0(x)|^r + \left| T \left[\frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} g(y) d\mu(y) \right] (x) \right|^r \right\} d\mu(x) \\ &\lesssim r \int_0^{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})} t^{r-1} \mu(\{x \in \mathcal{X} : |Tg_0(x)| > t\}) dt \\ &\quad + r \int_{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})}^\infty t^{r-1} \mu(\{x \in \mathcal{X} : |Tg_0(x)| > t\}) dt + \ell^r \\ &\lesssim \mu(\mathcal{X}) \int_0^{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})} t^{r-1} dt + \|g_0\|_{H^1(\mu)} \int_{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})}^\infty t^{r-2} dt + \ell^r \\ &\lesssim [\mu(\mathcal{X})]^{1-r} \|g_0\|_{H^1(\mu)}^r + \ell^r \lesssim \ell^r, \end{aligned}$$

which implies (3.7).

Observe that $\int_{\mathcal{X}} (|Tg|^r - F) d\mu(x) = 0$ and for any $R > 0$,

$$\sup_{0 < \ell < R} \ell^p \mu(\{x \in \mathcal{X} : N(|Tg|^r - F)(x) > \ell\}) \leq R^p \mu(\mathcal{X}) < \infty.$$

From this together with Lemma 3.2, $M_r^\sharp(F) = 0$, (3.7) and an argument similar to that used in Case (i), we conclude that there exists a positive constant \tilde{c} such that

$$\begin{aligned} \sup_{\ell > \ell_0} \ell^p \mu(\{x \in \mathcal{X} : N_r(Tf)(x) > \tilde{c}c_0\ell\}) &\leq \sup_{\ell > \ell_0} \ell^p \mu(\{x \in \mathcal{X} : N(|Tg|^r - F)(x) > (c_0\ell)^r\}) \\ &\quad + \sup_{\ell > \ell_0} \ell^p \mu(\{x \in \mathcal{X} : N_r(Th)(x) > c_0\ell\}) \\ &\lesssim \sup_{\ell > 0} \ell^p \mu(\{x \in \mathcal{X} : M_r^\sharp(Tg)(x) > c_0\ell\}) \\ &\quad + \sup_{\ell > 0} \ell^p \mu(\{x \in \mathcal{X} : N_r(Th)(x) > c_0\ell\}) \\ &\lesssim \sup_{\ell > 0} \ell^p \mu(\{x \in \mathcal{X} : N_r(Th)(x) > \ell\}) \lesssim \|f\|_{L^p(\mu)}^p, \end{aligned}$$

where in the first inequality we choose c_0 large enough such that $F \leq (c_0\ell)^r$. This completes the proof of Theorem 1.1. \square

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